Corrections to universal fluctuations in correlated systems: The two-dimensional *XY* model

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Generalized universality, as recently proposed, postulates a universal non-Gaussian form of the probability density function (PDF) of certain global observables for a wide class of highly correlated systems of finite volume N. Studying the two-dimensional XY model, we link its validity to renormalization group properties. It would be valid if there were a single dimension 0 operator, but the actual existence of several such operators leads to T-dependent corrections. The PDF is the Fourier transform of the partition function $Z(q)$ of an auxiliary theory which differs by a dimension 0 perturbation with a very small imaginary coefficient iq/N from a theory which is asymptotically free in the infrared. We compute the PDF from a systematic loop expansion of $\ln Z(q)$.

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Trying to understand universality properties of critical systems near a second order phase transition was extremely fruitful for the development of physics. It lead to the development of Wilsons renormalization group (RG) [1] which found applications ranging from solid state physics to elementary particle theory. Universality classes in the RG-sense depend on the dimension of the system and on the symmetry properties of the order parameter.

Recently a generalized universality has been proposed [2], which goes far beyond the known picture. It is therefore of great general interest to understand the conditions for its validity. Generalized universality is supposed to hold true for systems sharing the properties of strong correlations, finite volume, and self-similarity, no matter whether they have the same symmetries and dimensions nor whether they correspond to equilibrium or nonequilibrium systems. Such new universality is expressed in a non-Gaussian universal curve these systems share when the probability density function (PDF) of some global quantity of each system is plotted (universal fluctuations). Studying the two-dimensional (2D) XY model as an example, we link its validity to RG properties of the system, viz. the existence of dimension 0 perturbations of a system which is asymptotically free in the infrared. It would be exactly true if there were one single dimension 0 operator. But the actual existence of several dimension 0 operators in the model leads to T-dependent corrections to the supposedly universal curve which do not go away when the volume $N \mapsto \infty$. If there were no dimension 0 operator, the PDF would be Gaussian.

Let us recall some previous results. The supposed universal curve for the PDF, which we call BHP, is non-Gaussian in spite of the result that a naive application of the central limit theorem would have given; this possibility has been attributed [2] to the anomalous statistical property of finite size critical systems that, whatever their size is, cannot be divided into mesoscopic regions that are statistically independent, a necessary condition for the central limit theorem to apply. In Ref. [3] it was shown that a steady state in a closed turbulent flow experiment and a finite volume magnetic model, the 2D XY model in a finite square lattice, had the same universal non-Gaussian probability distribution function for the fluctuations of a global quantity. These are two examples of systems, among others (see [4–7]), that exhibit such data collapse.

In [8] it was argued that in finite-size systems that are in the critical regime standard scaling is a sufficient condition to exhibit data collapse. Nevertheless, as our results show, when one computes instead of the PDF more sensitive quantities (from the numerical point of view), such as its moments (skewness and kurtosis), temperature dependence shows up. This occurs within the range of temperatures of the expected data collapse.

A functional form for the BHP universal curve was suggested in [2] and it was also argued that this universal curve should be independent of both size and temperature of the system. This result was based on an earlier computation of the PDF of the magnetization of the 2D XY model in the harmonic (=spin-wave) approximation (the 2D HXY model) [9]. In this reference the PDF of the magnetization was carried out by resumming its moment series up to leading order in N (the system size), assuming that multiple loop diagrams can be neglected, and finally by performing an inverse Fourier integral.

Nevertheless, more recently [10] a numerical study of the full 2D XY model was performed, and the results suggested a small but systematic dependence of the PDF on the system temperature. This numerical analysis was performed by using high precision Monte Carlo simulations, which amounts to control the statistical independence of the configurations used to compute thermodynamical averages, as this system is affected by the well-known critical slowing down effect.

In this work we explain how to compute systematically

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the PDF for the 2D XY model within the loop expansion of an auxiliary theory with an imaginary coupling. It turns out that the one-loop approximation agrees with the curve BHP computed in Ref. [11]. But higher loop corrections have a different T-dependence. They are not suppressed by powers of $1/N$ and the T-dependence of their contribution to supposedly universal quantities such as skewness and kurtosis does not disappear for large N. This unexpected breakdown of strict BHP universality is not due to the corrections to the harmonic approximation to the Hamiltonian, but comes from the fact that the nonlinear order parameter (the magnetization) is a sum of several dimension 0 operators which contribute at different order of the loop expansion.

We will later confirm the known fact [11] that the corrections to the spin wave approximation will not affect the PDF, for temperatures well below the Berezinskii-Kosterlitz-Thouless critical temperature T_{BKT} and for large N, apart from a renormalization of temperature. Therefore the PDF for the XY and the HXY model have the same large N limit except for a renormalization of temperature, but neither of them obeys strict BHP-universality. Because vortices are neglected in perturbation theory, results are only valid well below T_c.

Let us consider a classical statistical mechanical system which lives on a lattice with N sites x, to which variables ϕ_x are attached, and with a Boltzmann weight $e^{-\beta\mathcal{H}(\phi)}D\phi$, where $D\phi=\Pi_x d\phi_x$ and $d\phi_x$ is a measure on the space in which ϕ_x takes its values. Let $A(\phi)$ be a global quantity of the form

$$A(\phi) = \frac{1}{N}\sum_x \Phi(\phi_x),$$

where $\Phi(\phi_x)$ is some function of ϕ_x. The magnetization per site in a ferromagnet is an example of such a global variable.

The PDF is defined as

$$P(M) = \frac{1}{Z^0}\int D\phi e^{-\beta\mathcal{H}(\phi)}\delta(A(\phi)-M) \tag{1}$$

with partition function $Z_0=\int D\phi e^{-\beta\mathcal{H}(\phi)}$. The mean $\langle M\rangle$ and mean square fluctuation $\sigma^2=\langle(M-\langle M\rangle)^2\rangle$ of A are computable from the PDF as $\langle M\rangle=\int MP(M)dM$ and $\sigma^2=\int(M-\langle M\rangle)^2P(M)dM$, respectively.

Inserting the Fourier representation of the δ-function, we have

$$P(M) = \int_{-\infty}^{\infty}\frac{dq}{2\pi}e^{-iq(M-\langle M\rangle)}Z(q), \tag{2}$$

where

$$Z(q) = \frac{e^{-iq\langle M\rangle}}{Z_0}\int D\phi \exp\left\{-\beta\mathcal{H}(\phi)+\frac{iq}{N}\sum_x \Phi(\phi_x)\right\}. \tag{3}$$

The hypothesis of BHP universality says that the PDF, considered as a function

$$P(M) = F\left(\frac{M-\langle M\rangle}{\sigma}\right)$$

of $(M-\langle M\rangle)/\sigma$, is a universal function F, the same for many strongly correlated critical systems, independent of temperature $T=\beta^{-1}$ and of system size N, provided N is large enough.

We see that the PDF is the Fourier transform of the "partition function" $Z(q)$, normalized to $Z(0)=1$, of an auxiliary theory whose Hamiltonian differs from that of the original theory by a perturbation with an imaginary coupling iq/N which is small of order $1/\text{volume}=N^{-1}$. On finite systems, $Z(q)$ will typically be a holomorphic function of q in a small neighborhood of zero, and $Z(q)$ is a true partition function for positive imaginary q. We choose to extract the factor $\exp(iq\langle M\rangle)$ from the partition function so that its derivative $Z'(0)=0$.

The hypothesis of BHP-universality asserts that for the above-mentioned systems, the "partition function" $Z(q)$ is a universal non-Gaussian function of $x=-\sigma q$. We consider the magnetization per site of the two-dimensional XY model. ϕ_x can be considered as real variables, and the Hamiltonian is

$$\mathcal{H}(\phi) = -J\sum_{\langle x,y\rangle}\cos(\phi_x-\phi_y). \tag{4}$$

Without loss of generality, the coupling constant $J>0$ can be set to 1. The sum runs over pairs of nearest neighbor sites x, y on a square lattice of side length $L=\sqrt{N}$. The Hamiltonian is invariant under a global rotation of the spins $(\cos\phi_x,\sin\phi_x)$ and under translations $\phi_x\mapsto\phi_x+2\pi n_x$, n_x integer. By a global shift of the variables ϕ_x one can achieve $\Sigma_x\phi_x=0$. The magnetization per site is then given by

$$\Phi(\phi_x) = \cos\phi_x. \tag{5}$$

The density $\cos\phi_x$ is a sum of dimension 0 perturbations which are eigenmodes of the linearized renormalization group. The existence of a dimension 0 perturbation explains why the q-dependent perturbation can make a contribution of order N^0 to $\ln Z(q)$, in each order q^n, despite the small coefficient $\propto N^{-1}$. The existence of several (in fact infinitely many) dimension 0 perturbations explains why there is no strict BHP-universality. If the perturbation had dimension >0, successive orders in q would be suppressed by negative powers of N, so that the term $\propto q^2$ would dominate after the term $\propto q$ is subtracted from $\ln Z(q)$, and the PDF would be Gaussian.

The existence of several dimension 0 operators is unusual even in two space dimensions. But for the 2D XY model it follows from features of free-field theory because the (known) accuracy of the harmonic approximation to the Hamiltonian implies that we consider RG-transformations in the vicinity of a Gaussian fix point.

In the spin wave approximation, the Hamiltonian is approximated by

$$\mathcal{H}_{SW}(\phi) = \frac{1}{2}\sum_x \sum_{\mu=1}^{2} |\nabla_\mu \phi(x)|^2, \tag{6}$$

where ∇_μ is the finite difference derivative. This is the Hamiltonian of the *HXY* model.

Since $\beta = 1/T$ appears as a factor in front of $\mathcal{H}$, the logarithm of $Z(q)$ can be computed as a power series in T, for fixed qT/N. This is called the loop expansion since the parameter T plays the role of $\hbar$ in field theory [12]. The expansion up to n loops includes terms up to order T^{n-1}.

On finite lattices, $\ln Z(q)$ may be expanded in a power series in q.

$$iq\langle M\rangle + \ln Z(q) = iq + \sum_{n\geq 1} (-iqT)^n \alpha_n(T). \tag{7}$$

The coefficients α_n also depend on system size N. By definition, $Z(q)$ is normalized to 1 and obeys $Z'(0)=0$, hence

$$1 - \langle M\rangle = T\alpha_1(T) \tag{8}$$

and the mean square fluctuation $\sigma^2 = -d^2/d^2q^2 \ln Z(q=0)$ equals

$$\sigma^2 = 2T^2\alpha_2(T). \tag{9}$$

To obtain the PDF with normalized first and second moments, one considers $\ln Z(q)$ as a function of $x=-q\sigma$. It has the form

$$-iq(M - \langle M\rangle) + \ln Z(q) = ix\frac{M-\langle M\rangle}{\sigma} - \frac{x^2}{2} + \Psi(x),$$

with $\Psi(x) = O(x^3)$, and the higher moments are obtained from it as

$$\left\langle \left(\frac{M-\langle M\rangle}{\sigma}\right)^n \right\rangle = \left(i\frac{d}{dx}\right)^n \exp\left(-\frac{1}{2}x^2 + \Psi(x)\right)_{x=0}, \tag{10}$$

for $n \geq 3$. Inserting the power series expansion (7) of $\ln Z$ into Eq.(2) results in

$$P(M) = \int_{-\infty}^{\infty} \frac{dx}{2\pi\sigma} \exp\left\{ ix\frac{(M-\langle M\rangle)}{\sigma} - \frac{1}{2}x^2 + \sum_{n\geq 3} \alpha_n(T)\left(\frac{ixT}{\sigma}\right)^n \right\}. \tag{11}$$

The skewness $s = \langle (M-\langle M\rangle)^3\rangle \sigma^{-3}$ and kurtosis $c = \langle (M - \langle M\rangle)^4\rangle\sigma^{-4}$ are obtained from Eq. (10) as

$$s = -6\left(\frac{T}{\sigma}\right)^3 \alpha_3, \tag{12}$$

$$c = 3\left\{1 + 8\left(\frac{T}{\sigma}\right)^4 \alpha_4\right\}. \tag{13}$$

BHP-universality would require that the coefficients $\alpha_n(T)(T/\sigma)^n$ of $(ix)^n$ have universal values, given by the BHP-formula $g_n(\frac{1}{2}g_2)^{-n/2}/2n$ which we recover below as the one-loop approximation. In particular, these coefficients should be T-independent, and also the skewness and kurtosis should be T-independent. We will compute the coefficients α_n to two-loop order.

We extract the quadratic terms

$$\beta\mathcal{H}(\phi) = \beta\mathcal{H}_{SW}(\phi) + V_1(\phi), \tag{14}$$

$$V_1(\phi) = -\beta\sum_x \left(\frac{1}{4!}(\nabla\phi_x)^4 + \cdots\right), \tag{15}$$

$$\frac{iq}{N}\sum_x \cos\phi_x = \frac{iq}{N}\sum_x\left(1 - \frac{1}{2}\phi_x^2\right) - V_2(\phi_x), \tag{16}$$

and define the q-dependent propagator

$$\Gamma = \left(I + \frac{iqT}{N}G\right)^{-1} G, \tag{17}$$

where

$$G(x) = N^{-1}\sum_{\mathbf{k}\neq 0} e^{-ikx}\tilde{G}(\mathbf{k}), \tag{18}$$

$$\tilde{G}(\mathbf{k}) = \frac{1}{\mathbf{K}^2}, \tag{19}$$

$$\mathbf{K}^2 = \sum_{\mu=1,2} \sin^2(k_\mu/2) \tag{20}$$

is the propagator of a massless scalar free field on a finite two-dimensional lattice with the zero mode removed.

Using the familiar device of introducing a source term $\langle j,\phi\rangle$, the integral defining $Z(q)$ can be formally performed [13] with the result

$$Z(q) = \frac{1}{Z_0} e^{-iq(\langle M\rangle -1)} \exp\left\{\frac{1}{2}\mathrm{Tr}\ln[T\Gamma]\right\} \times \exp\left\{-V\left[\frac{\delta}{\delta j}\right]\right\} \exp\left\{\frac{T}{2}\langle j, \Gamma j\rangle\right\}_{j=0}, \tag{21}$$

with $V = V_1 + V_2$.

In one-loop approximation, we obtain from Eq. (21)

$$-iq(M-\langle M\rangle) + \ln Z(q) = -iq(M-1) + \frac{1}{2}\mathrm{Tr}\ln[\Gamma]$$

plus q-independent terms. Upon inserting the definition (17) of Γ,

$$\mathrm{Tr}\ln\Gamma = \sum_{\mathbf{k}\neq 0}\left\{\ln\tilde{G}(\mathbf{k}) - \ln\left(1 + \frac{iqT}{N}\tilde{G}(\mathbf{k})\right)\right\} \tag{22}$$

the PDF in one-loop approximation can be computed by numerical Fourier transformation according to Eq. (2). A similar computation is also possible when the two-loop correction is added; in harmonic approximation it requires

$$\Gamma(0) = \frac{1}{N}\mathrm{Tr}\,\Gamma = \frac{1}{N}\sum_{\mathbf{k}\neq 0}\left(1 + \frac{iqT}{N}\tilde{G}(\mathbf{k})\right)^{-1}\tilde{G}(\mathbf{k})$$

according to Eq. (27) below.

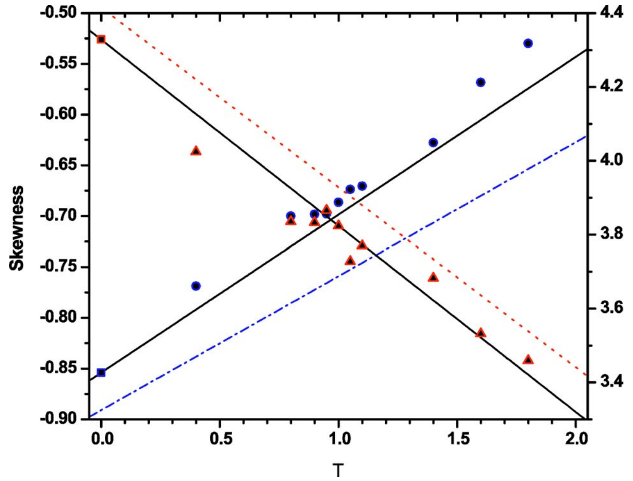

FIG. 1. Skewness and kurtosis. Full triangles and circles are Monte Carlo data for a lattice with $N=16^2$ points. Full lines correspond to the numerical values given in Eqs. (36) and (38). The exact one-loop results at $T=0$ are shown as full square dots. Dotted and dash-dotted lines correspond to the thermodynamic limit ($N=\infty$).

This result of the PDF in one-loop approximation reproduces exactly the previous result [2,11] for the BHP PDF. In Sec. II A of [11] it is stated "It appears that the values of multiple loop graphs, like the first one in Fig. 1(c), are zero in the thermodynamic limit." But in fact this is not true for all graphs, and as a result the BHP PDF is merely a lowest order approximation which becomes exact in the limit $T\mapsto 0$ only.

We will compute the two-loop contribution below and comment on the precise relation between the diagrammatic expansion here and in [11] later on.

Following Ref. [2] we define the quantities g_n via powers of G as

$$G^n(0) = N^{-1}\,\mathrm{Tr}\,G^n = N^{n-1}g_n. \tag{23}$$

g_1 is logarithmically divergent in the thermodynamic limit, $g_1 \sim 1/4\pi \ln CN$, $C=1.8456$ [11] while the remaining ones become N-independent for large enough N and in the thermodynamic limit they read

$$g_2 = 0.38667\times 10^{-2}, \quad g_3 = 0.7572\times 10^{-4},$$
$$g_4 = 1.763\times 10^{-6}.$$

Expanding the q-dependent propagator in powers of q we obtain the one-loop approximation $\alpha_n^{(1)}(T)$ of the coefficients $\alpha_n(T)$, viz.

$$\alpha_n^{(1)}(T) = \frac{g_n}{2n} \tag{24}$$

together with the results for the expectation value $\langle M\rangle$ of the magnetization and its mean square fluctuation σ^2 in one-loop approximation

$$\langle M\rangle^{(1)} = 1 - Tg_1/2, \tag{25}$$

$$(\sigma^{(1)})^2 = T^2 g_2/2. \tag{26}$$

Note that σ is proportional to T, so that the factors $(T/\sigma)^k$ are T-independent in one-loop approximation.

Inserting the result (24) into Eq. (11) we recover the known formula of BHP for the universal PDF.

Numerically, the skewness $s=\langle(M-\langle M\rangle)^3\rangle\sigma^{-3}$ and kurtosis $c=\langle(M-\langle M\rangle)^4\rangle\sigma^{-4}$ in one-loop approximation are obtained from Eq. (10) as $s=0.8907$ and $c=4.4150$ for $N\mapsto\infty$.

In two-loop approximation there are two corrections, $\alpha_n = \alpha_n^{(1)} + \alpha_n^{(2harm)} + \alpha_n^{(2anh)}$. The first correction comes from the ϕ_x^4 contribution to $iq/N\cos\phi_x$. Its contribution to $-iq(M-\langle M\rangle)+\ln Z(q)$ is

$$\frac{iqT^2}{8}[\Gamma(0)]^2. \tag{27}$$

Expanding Γ in powers of q gives

$$\alpha_k^{(2harm)} = -\frac{T}{8}\sum_{m=0}^{k-1} g_{m+1}g_{k-m}. \tag{28}$$

This term is also present in the HXY model.

The second correction comes from the anharmonic term $-\beta(\nabla\phi_x)^4/4!$ in the Hamiltonian. It is absent in the HXY model. It evaluates to

$$\frac{T}{8}N([\Delta\Gamma(0)]^2 - [\Delta G(0)]^2). \tag{29}$$

The q-independent subtraction $\propto[\Delta G(0)]^2$ restores the normalization $\ln Z(0)=0$. Expanding in powers of q and including terms of order $1/N$ results in

$$\alpha_k^{(2anh)} = \frac{T}{4}\left(g_k + \frac{1}{2N}\sum_{l=1}^{k-1} g_{k-l}g_l\right). \tag{30}$$

We shall argue below that to leading order in $1/N$ this anharmonic correction only amounts to a renormalization of temperature which cannot lead to a violation of BHP-universality. The presence of the $1/N$-corrections may explain why, for finite lattice sizes, the shape of the PDF of the full 2D XY model displays a stronger temperature dependence than the spin-wave approximation, as shown in Ref. [10].

In two-loop approximation, to next to leading order in T, the average magnetization, mean square fluctuation, skewness s, and kurtosis c including anharmonic corrections come out as

$$\langle M\rangle = 1 - \frac{T}{2}g_1 - \frac{T^2}{4}g_1 + \frac{T^2}{8}g_1^2, \tag{31}$$

$$\sigma^2 = T^2\frac{g_2}{2}\left\{1 - Tg_1 + T\left(1 + \frac{1}{2N}\frac{g_1^2}{g_2}\right)\right\}, \tag{32}$$

$$s = -g_3\left(\frac{2}{g_2}\right)^{3/2}\left\{1 + T\left(-\frac{3}{4}\frac{g_2^2}{g_3} + \frac{3}{2N}\left[\frac{g_1g_2}{g_3} - \frac{g_1^2}{2g_2}\right]\right)\right\}, \tag{33}$$

$$c = 3\left\{1 + \frac{4g_4}{g_2^2} + \frac{4T}{g_2^2}\left(-2g_2g_3 + \frac{1}{N}\left[2g_1g_3 + g_2^2 - \frac{g_1^2 g_4}{g_2}\right]\right)\right\}. \quad (34)$$

Using the values of g_n for the appropriate volumes N, skewness and kurtosis for the XY model evaluate to

$$s = -0.891 + 0.1319T \ (N = \infty) \quad (35)$$

$$= -0.854 + 0.1555T \ (N = 16^2), \quad (36)$$

$$c = 4.415 - 0.470T \ (N = \infty) \quad (37)$$

$$= 4.328 - 0.504T \ (N = 16^2). \quad (38)$$

As stated above, terms suppressed by powers of N^{-1} come from the anharmonic corrections. We see that both s and c have a T-dependence which does not disappear as $N \mapsto \infty$. This disproves strict BHP-universality. The values for $N=\infty$ and $N=16^2$ are close to each other, yet different enough to suggest that comparison with Monte Carlo data at $N=16^2$ will yield a sensitive test of the accuracy of the calculation.

In Fig. 1 we show Monte Carlo results, similar to those of Ref. [10], for the skewness and kurtosis in the temperature range $T=0.2,\ldots,1.8$ together with the $T=0$ result given by the one-loop approximation, and the two-loop results (36) and (38) up to order T. The agreement is quite satisfactory.

In one-loop approximation, the $(iq/N)\cos\phi_x$-perturbation is approximated by $(iq/N)(1-\phi_x^2/2)$. The two-loop correction takes the ϕ_x^4-term in the Taylor expansion of $\cos\phi_x$ into account. The different T and x-dependence of the two-loop correction to $\ln Z(q=-x/\sigma)$ can therefore be attributed to the fact that it comes from a different relevant operator of dimension 0.

We will now argue that the effect of the corrections to the spin wave approximation on the PDF is merely a renormalization of temperature, up to corrections of order N^{-1}, viz.

$$T \mapsto T(1 + T/2 + \cdots).$$

This comes from the fact that the theory with Hamiltonian $\mathcal{H}$ is asymptotically free in the infrared.

The one-loop approximation is not affected by the renormalization of temperature because it is T-independent, and the two-loop correction is also unaffected to order T because the correction would be of order T^2. Thus, to order T, the skewness and kurtosis in the XY and the HXY models differ only by corrections proportional to $1/N$.

Because the coefficient of the q-dependent perturbation in $Z(q)$ is of order N^{-1}, the q-dependent part of $\ln Z(q)$ is entirely determined by the infrared behavior of the theory, up to corrections of order N^{-1}. Therefore it could be computed from an effective low energy theory. In principle its Hamiltonian would have to be computed by a real space renormalization group (RG) calculation [1]. Since V_1 is a sum of irrelevant perturbations of massless free-field theory, their coefficients get suppressed when the length scale gets enlarged, but they can give rise to a renormalization of the coefficient T^{-1} of the marginal $|\nabla\phi_x|^2$ term. This is the aforementioned renormalization of temperature. In principle there could also be a renormalization of the coefficients of the relevant perturbation, i.e., of the coefficient q of ϕ^2 in one-loop approximation. But a multiplicative renormalization of q drops out when the second moment of the PDF is normalized. New terms in the perturbed effective action like $T^3\cos 2\phi_x$ can appear in higher orders, but they are too small to be detectable. Note, however, that the perturbative treatment of the corrections to the spin wave approximation does not take vortices into account. Its results are therefore only valid well below T_c.

To verify the RG-argument, we inspect the anharmonic corrections (30). The addition of $Tg_k/4$ to $\alpha_k^{(1)}=g_k/2k$ corrects the one-loop result by the substitution

$$g_k \mapsto g_k(1 + kT/2) \quad (39)$$

to first order in T. Clearly, this has no effect on $g_k\sigma^{-k}$ to order T, and therefore the normalized PDF remains unchanged.

Next we discuss the precise relation between our loop expansion and the diagrammatic expansion of Ref. [11], and hyperscaling.

One may envisage generalization of the kinetic term

$$\frac{\beta}{2}\sum_x \sum_\mu |\nabla_\mu \phi_x|^2 = -\frac{\beta}{2}\langle\phi, \Delta\phi\rangle = \frac{1}{2T}\langle\phi, G^{-1}\phi\rangle$$

by admitting arbitrary positive G, i.e., arbitrary propagator $G(x,y)$. [In the translation invariant case, the propagator is considered as a function $G(x-y)$ of a single variable.] Diagrammatic expressions exhibit results in their dependence on G, as sums of products of propagators $G(x,y)$. Therefore they have a universal meaning, and the diagrammatic expansion for the *same* quantity must agree, no matter how the computation is organized. Our way of organizing the expansion is more economical in that it avoids having to go through the moments; they are here summed from the start and $Z(q)$ is their generating function. We obtain diagrammatic expansions for (connected parts $\langle M^p\rangle^c$ of) $\langle M^p\rangle$ by expanding our q-dependent propagator Γ and $\mathrm{Tr}\ln\Gamma$ in powers of q, cf. Eqs. (17), (22), and (23). To recover the diagrammatic expansion of Ref. [11] for $\langle M^p\rangle/\langle M\rangle^p$ one must divide by $\langle M\rangle^p$. As shown in the Appendix, the loop expansion can be reorganized so that $\langle M\rangle$ appears as a factor multiplying q. This implies that the diagrams for $\langle M^p\rangle/\langle M\rangle^p$ are precisely the diagrams for $\langle M^p\rangle$ which contain no tadpole. A tadpole is a line in the graph whose source and target agree; tadpoles produce factors $TG(0)/2=Tg_1/2$. Figure 2 shows the connected two-loop graphs without tadpoles which contribute for $p=3,4$, in spin wave approximation.

For the sake of comparison let us evaluate these diagrams.

We remember that in a scalar field theory with interaction $V=\lambda/n!\phi^n$, vertices have n incident lines and contribute a factor $-\lambda$. The $-$sign comes because the Boltzmannian involves e^{-V}. In the expansion of $iq/N\cos\phi_x$, even powers ϕ_x^n appear with factors $1/n!$ and alternating sign. The Feynman rules for our theory in harmonic approximation say that $iq\langle M\rangle + \ln Z(q)$ is the sum of all connected vacuum diagrams, and in these diagrams there is

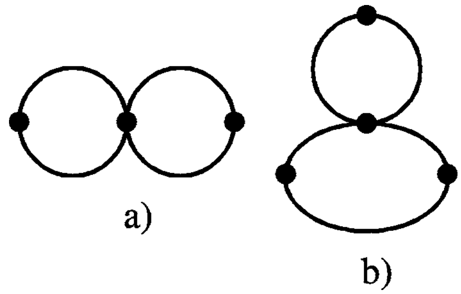

FIG. 2. Connected two-loop graphs in spin wave approximation. (a) Contribution to $i^3\langle M^3\rangle^c/3! = (-iqT)^3\alpha_3$, and (b) contribution to $i^4\langle M^4\rangle^c/4! = (-iqT)^4\alpha_4$.

(1) a factor $iq/N(-)^{n/2}$ for every vertex with an even number n of incident lines;

(2) a factor $TG(x-y)$ for a line from a vertex at x to a vertex at y;

(3) a sum over x for every vertex; and

(4) a symmetry factor $1/g$, where g is the number of elements of the symmetry group of the graph.

Since every vertex contributes a factor q, the vacuum diagrams with k vertices sum to the summand $(-iqT)^k\alpha_k(T)$ in the expansion (7). Contributions with tadpoles can be eliminated by setting $g_1=0$. The symmetry factor for the graph (a) is $\frac{1}{8}$, and the symmetry factor for the graph (b) is $\frac{1}{4}$. Using that $\Sigma_y G(x-y)G(y-x) = G^2(0) = Ng_2$, etc., the diagrams evaluate to results in agreement with Eq. (28), viz.

$$(-iqT)^3\alpha_3^{2harm} = (iq)^3\frac{T^4}{8}g_2^2 + O(g_1), \tag{40}$$

$$(-iqT)^4\alpha_4^{2harm} = -(iq)^4\frac{T^5}{4}g_2g_3 + O(g_1). \tag{41}$$

We see that the contribution of the diagrams shown in Fig. 2 does not vanish in the thermodynamic limit.

The result that $\langle M\rangle$ can be factored out confirms that hyperscaling remains true to all orders in the loop expansion. Hyperscaling asserts that $\langle M\rangle^2/\sigma^2$ has a thermodynamic limit. Within the framework of a loop expansion, this must be true order by order in T. This is not trivially true because g_1 has no thermodynamical limit. Hyperscaling also remains true in the presence of anharmonic corrections because these merely renormalize the temperature, as we saw. Our two-loop results for σ and $\langle M\rangle$ which include anharmonic corrections give

$$\sigma = \sqrt{\frac{g_2}{2}}T\langle M\rangle\left(1 + \frac{1}{2}T + O(T^2)\right) \tag{42}$$

for large N, which should be compared with Eq. (19) of Ref. [11].

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APPENDIX: NORMAL ORDERED PROCEDURE

We introduce the normalized Gausssian measure with covariance TG

$$d\mu_{TG}(\phi) = \frac{1}{Z_0}\exp\left\{-\frac{\beta}{2}\sum|\nabla_\mu\phi_x|^2\right\}D\phi. \tag{A1}$$

It has the characteristic function

$$\int d\mu_{TG}(\phi)e^{i\langle j,\phi\rangle} = e^{-T/2\langle j,Gj\rangle}. \tag{A2}$$

If $A(\phi)$ is a function of ϕ which is integrable with respect to the Gaussian measure, it can be written in normal ordered form [14]. The normal ordering operation is typically indicated by $::$, but it depends on the covariance of the Gaussian measure.

$$A(\phi) = :B(\phi):, \tag{A3}$$

$$B(\phi) = \int d\mu_{TG}(\xi)A(\phi+\xi). \tag{A4}$$

In particular

$$e^{i\alpha\phi_x} = e^{-1/2\alpha^2TG(0)}:e^{i\alpha\phi_x}:. \tag{A5}$$

It follows that

$$\cos\phi_x = e^{-1/2TG(0)}:\cos\phi_x: \tag{A6}$$

$$\int d\mu_{TG}(\phi):\cos\phi_x := 1. \tag{A7}$$

Under general conditions, B is a holomorphic function of ϕ which can be expanded into a power series. In particular

$$:\cos\phi_x := 1 - \frac{1}{2}:\phi_x^2: + \frac{1}{4!}:\phi_x^4: + \cdots. \tag{A8}$$

It is well known that the normal products: ϕ_x^p: are always eigenmodes of the linearized renormalization group at the Gaussian fix point determined by TG. In two dimensions, with fix point Hamiltonian $\beta\mathcal{H}_{SW} = \frac{1}{2}\beta\Sigma|\nabla_\mu\phi|^2$ they all have dimension 0.

Consider now the partition function

$$\tilde{Z}(q) = e^{iq\langle M\rangle}Z(q). \tag{A9}$$

It follows from Eq. (2) that the moments

$$\langle M^p\rangle = \left(-i\frac{\partial}{\partial q}\right)^p\tilde{Z}(q)|_{q=0}. \tag{A10}$$

We restrict our attention to the spin wave approximation, replacing $\mathcal{H}$ by $\mathcal{H}_{SW}$. Inserting Eq. (A6) into the definition (3) of $Z(q)$, we obtain

$$\tilde{Z}(q) = \int d\mu_{TG}(\phi)\exp\left\{\frac{iq}{N}e^{-T/2G(0)}\sum_x:\cos\phi_x:\right\}. \tag{A11}$$

From this and Eq. (A7) we recover the known result for the magnetization in spin wave approximation,

$$\langle M \rangle = e^{-T/2G(0)}. \tag{A12}$$

Therefore

$$\tilde{Z}(q) = \hat{Z}(\langle M \rangle q) \tag{A13}$$

with a new partition function

$$\hat{Z}(q) = \int d\mu_{TG}(\phi) \exp\left\{ \frac{iq}{N} \sum_x :\cos \phi_x: \right\}, \tag{A14}$$

and

$$\langle M^p \rangle / \langle M \rangle^p = \left(- i \frac{\partial}{\partial q} \right)^p |\hat{Z}(q)|_{q=0}. \tag{A15}$$

The partition function $\hat{Z}(q)$ is exactly like $\tilde{Z}(q)$ except for the normal ordering of $\cos \phi_x$. In a diagrammatic expansion, normal ordering eliminates tadpoles. The diagrams which contribute to $\langle M^p \rangle / \langle M \rangle^p$ are therefore exactly the diagrams without tadpoles which contribute to $\langle M^p \rangle$. It is more convenient to consider truncated expectation values

$$\langle M^p \rangle^c / \langle M \rangle^p = \left(- i \frac{\partial}{\partial q} \right)^p \ln \hat{Z}(q)|_{q=0}. \tag{A16}$$

They have expansions in connected diagrams. The nth order in the loop expansion for $\ln \hat{Z}$ sums connected diagrams with n loops. In one-loop approximation only the term $1 - \frac{1}{2} : \phi_x^2 :$ in the expansion (50) of: $\cos \phi_x$: contributes, but the $1/4! : \phi_x^4 :$ term begins to contribute in two-loop order, and so on.

[1] K. Wilson, Phys. Rev. B **4**, 3174 (1971).
[2] S. T. Bramwell *et al.*, Phys. Rev. Lett. **84**, 3744 (2000).
[3] S. T. Bramwell *et al.*, Nature (London) **396**, 552 (1998).
[4] P. Bak, C. Tang, and K. Weisenfeld, Phys. Rev. Lett. **59**, 381 (1997).
[5] K. Sneppen, Phys. Rev. Lett. **69**, 3538 (1992).
[6] P. Sinha-Ray, L. Borda de Agua, and H. J. Jensen, Physica D **157**, 186 (2001).
[7] E. Caglioti, V. Loreto, H. Hermann, and M. Nicodemi, Phys. Rev. Lett. **79**, 1575 (1997).
[8] V. Aji and N. Goldenfeld, Phys. Rev. Lett. **86**, 1007 (2001).
[9] P. Archambault *et al.*, J. Appl. Phys. **83**, 7234 (1998).
[10] G. Palma, T. Meyer, and R. Labbé, Phys. Rev. E **66**, 026108 (2002).
[11] S. T. Bramwell *et al.*, Phys. Rev. E **63**, 041106 (2001).
[12] G. Palma, Z. Phys. C **54**, 679 (1992).
[13] K. Symanzik, J. Math. Phys. **7**, 510 (1966).
[14] J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View* (Springer, New York, 1987).